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# Furstenberg Theorem for Frequently Hypercyclic Operators

**Eunsang Kim**

Department of Applied Mathematics
Hanyang University
Ansan Kyunggi-do, Korea

**Tae Ryong Park**[1]

Department of Computer Engineering
Seokyeong University
Seoul, Korea



**Abstract**

In this paper, we show that if the direct sum $T \oplus T$ of frequently hypercyclic operators is frequently hypercyclic, then every higher direct sum $T \oplus \cdots \oplus T$ is also frequently hypercyclic.

**Mathematics Subject Classification:** 47A16

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## 1 Introduction

In this paper, we study the dynamics of linear operators on a separable $F$-space $X$. A bounded linear operator $T$ on $X$ is said to be hypercyclic if there

[1]corresponding author

is a vector $x \in X$ such that the orbit $O(x,T) = \{T^n x \mid n \in \mathbb{N}\}$ is dense in $X$. The operator $T$ is said to be topologically transitive if, for every pair of non-empty open subsets $U$ and $V$, there is an integer $n$ such that $T^n U \cap V \neq \emptyset$. By the Baire category theorem, topological transitivity of $T$ is equivalent to the hypercyclicity of $T$. See [9] and [4] for details and references. If $T \oplus T$ is hypercyclic, then the operator $T$ is called weakly mixing. It is shown in [5] that the weakly mixing property is equivalent to the Hypercyclicity Criterion. On the other hand, as shown in [7] and [2], hypercyclic operators may not be weakly mixing, see also [3]. An interesting fact is so-called the Furstenberg theorem, which is given as follows: if $T$ is weakly mixing, then the $n$-fold product is $T \times \cdots \times T$ is weakly mixing for $n \geq 2$. The proof is given in [9] by using the 4-set trick. In the linear setting we have

**Theorem 1.1.** *Let $X$ be a separable $F$-space. If $T \oplus T$ is hypercyclic, then the higher sum $T \oplus \cdots \oplus T$ is also hypercyclic.* □

The $T$-orbit of a hypercyclic operator visits each non-empty open subsets of $X$. Then it is natural to ask how often the orbit visits each non-empty open sets in $X$ and it leads to the notion of the frequently hypercyclic operators which has been introduced by Bayart and Grivaux, see [1] and [6]. In [8], it is shown that every frequently hypercyclic operator is weakly mixing. Based on ideas given in [8], we prove the Furstenberg theorem for the frequently hypercyclic operators.

# 2 Frequently Weakly Mixing Operators

Let $X$ be a separable $F$-space and let $\mathcal{L}(X)$ be the space of continuous linear operators on $X$. By definition, an operator $T \in \mathcal{L}(X)$ is hypercyclic if there is a vector $x \in X$ such that the orbit

$$O(x,T) = \{T^n x \mid n \in \mathbb{N}\}$$

is dense in $X$. In other words, the $T$-orbit $O(x,T)$ intersects with each non-empty open set $U$ in $X$. For a non-empty open subset $U$ of $X$, define

$$\mathbf{N}(x,U) = \{n \in \mathbb{N} \mid T^n x \in U\}.$$

If an operator $T$ on $X$ is hypercyclic, then there is a vector $x \in X$ such that for each non-empty open set $U$ in $X$, the set $\mathbf{N}(x,U)$ are all non-empty. For any non-empty open sets $U$ and $V$, let us define the *return set* as follows:

$$\mathbf{N}(U,V) = \{n \in \mathbb{N} \mid T^n U \cap V \neq \emptyset\}.$$

By the topological transitivity, if $T$ is hypercyclic, then each set $\mathbf{N}(U,V)$ is non-empty.

If $T$ is weakly mixing, then there is a natural number $n$ such that for each open subsets $U_1, U_2$ and $V_1, V_2$ of $X$ such that

$$T^n U_1 \cap V_1 \neq \emptyset \quad \text{and} \quad T^n U_2 \cap V_2 \neq \emptyset.$$

Then the $T \in \mathcal{L}(X)$ is weakly mixing if and only if

$$\mathbf{N}(U_1, V_1) \cap \mathbf{N}(U_2, V_2) \neq \emptyset. \tag{1}$$

See [8], [9] and [4] for other formulas which are equivalent to (1).

The frequently hypercyclicity corresponds to the largeness of each sets $\mathbf{N}(x, U)$, in other words, how frequently the $T$-orbit intersects with each open set $U$. Let us first recall that the lower density of a subset $A$ in $\mathbb{N}$ which is given by

$$\underline{\text{dens}}(A) = \liminf_{N \to \infty} \frac{|A \cap [1, N]|}{N}$$

where $|A \cap [1, N]|$ denotes the cardinality of the set $A \cap [1, N]$.

**Definition 2.1.** Let $X$ be a topological vector space and let $T \in \mathcal{L}(X)$. The operator $T$ is called *frequently hypercyclic* if there is a vector $x \in X$ such that for every non-empty open set $U$, $\mathbf{N}(x, U)$ has positive lower density. Such a vector $x$ is called frequently hypercyclic for $T$ and the set of all frequently hypercyclic vectors for $T$ is denoted by $FHC(T)$. When $T \oplus T$ is frequently hypercyclic, we will say that the operator $T$ is *frequently weakly mixing.* □

If we enumerate an infinite set $A \subset \mathbb{N}$ as an increasing sequence $(n_k)_{k \in \mathbb{N}}$, then it is easy to see that $A$ has positive lower density if and only if there is a constant $C$ such that

$$n_k \leq Ck \qquad \text{for all } k \geq 1$$

Thus, a vector $x \in X$ is frequently hypercyclic for $T$ if and only if for each non-empty open subset $U$ of $X$, there is a strictly increasing sequence $(n_k)$ and some constant $C$ such that

$$T^{n_k} x \in U \quad \text{and} \quad n_k \leq Ck$$

for all $k \in \mathbb{N}$. In other words, $n_k = O(k)$ as $k \to \infty$. First, we give a basic step for the Furstenberg theorem for frequently hypercyclic operators. Then the general form of the theorem is followed by the induction.

**Theorem 2.2.** *Let $X$ be a separable $F$-space and let $T \in \mathcal{L}(X)$. If $T$ is frequently weakly mixing, then 3-fold sum $T \oplus T \oplus T$ is frequently hypercyclic.*

*Proof.* We will show that there is a vector $x_1 \oplus x_2 \oplus x_3 \in X \oplus X \oplus X$ satisfying the following property: for each non-empty open subsets $U_1$, $U_2$ and

$U_3$ of $X$, there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers and a constant $C$ such that for $i = 1, 2, 3$,

$$T^{n_k} x_i \in U_i \quad \text{and} \quad n_k \leq Ck \quad \text{for all } k \in \mathbb{N}$$

First, we note that if $T$ is frequently weakly mixing, then $T$ is weakly mixing. By the Furstenberg theorem $T \oplus T \oplus T$ is also hypercyclic. Thus there is a hypercyclic vector $x_1 \oplus x_2 \oplus x_3 \in X \oplus X \oplus X$ for $T \oplus T \oplus T$. In other words, for each non-empty open subsets $U_1$, $U_2$ and $U_3$ of $X$, there is an integer $n$ such that $(T \oplus T \oplus T)^n(x_1 \oplus x_2 \oplus x_3) \in U_1 \oplus U_2 \oplus U_3$. Thus we have

$$\mathbf{N}(x_1, U_1) \cap \mathbf{N}(x_2, U_2) \cap \mathbf{N}(x_3, U_3) \neq \emptyset.$$

Suppose that $x_1 \oplus x_2$ is a frequently hypercyclic vector for $T \oplus T$. Then for each pair of non-empty open sets $U_1$ and $U_2$, there is a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ and a constant $C_1$ such that

$$T^{m_k} x_1 \in U_1, \quad T^{m_k} x_2 \in U_2 \quad \text{and} \quad m_k \leq C_1 k.$$

Since $T$ is hypercyclic, the return set $\mathbf{N}(U_1, U_2) = \{l \in \mathbb{N} \mid T^l U_1 \cap U_2 \neq \emptyset\}$ is non-empty and the $T$-orbit $O(x_1, T)$ is dense in $X$. Thus, there is an increasing sequence $(b_j)_{j\in\mathbb{N}}$ such that

$$x_2 = \lim_{j\to\infty} T^{b_j} x_1. \tag{2}$$

Since $T$ is continuous, for each $k \in \mathbb{N}$,

$$T^{m_k} x_2 = \lim_{j\to\infty} T^{b_j} T^{m_k} x_1 \in U_2. \tag{3}$$

Then there is an integer $N$ such that for all $j \geq 1$

$$T^{b_{j+N}} T^{m_k} x_1 \in U_2.$$

Thus, for each $k$, by taking $j = k$, we have

$$T^{b_{k+N}} T^{m_k} x_1 \in U_2 \quad \text{and} \quad T^{b_{k+N}} U_1 \cap U_2 \neq \emptyset. \tag{4}$$

In other words, the sequence $(b_{k+N})_{k\in\mathbb{N}}$ is in $\mathbf{N}(U_1, U_2)$. Since $x_1$ and $x_2$ are frequently hypercyclic, one can see that there is a constant $M$ such that $b_{k+N} \leq Mk$ for all $k \geq 1$ (*cf.* [3]). Let

$$U_1' = U_1 \cap T^{-b_{k+N}} U_2.$$

Then $\mathbf{N}(x_1, U_1') \subset \mathbf{N}(x_1, U_1)$. If $l \in \mathbf{N}(x_1, U_1')$ then for all $k \in \mathbb{N}$,

$$T^l x_1 \in U_1 \quad \text{and} \quad T^{b_{k+N}} T^l x_1 = T^l T^{b_{k+N}} x_1 \in U_2.$$

By (3) and (4), we get $l \in \mathbf{N}(x_2, U_2)$. Thus

$$\mathbf{N}(x_1, U_1') \subset \mathbf{N}(x_1, U_1) \cap \mathbf{N}(x_2, U_2). \tag{5}$$

Note that since the sequence $(b_{k+N})$ is of order $O(k)$, if the sets $\mathbf{N}(x_1, U_1)$ and $\mathbf{N}(x_2, U_2)$ are enumerated by strictly increasing sequences of order $O(k)$, respectively, then the subset $\mathbf{N}(x_1, U_1')$ is also enumerated by a strictly increasing sequence of order $O(k)$.

By applying the same argument for $\mathbf{N}(x_2, U_2) \cap \mathbf{N}(x_3, U_3)$, we obtain

$$\mathbf{N}(x_2, U_2') \subset \mathbf{N}(x_2, U_2) \cap \mathbf{N}(x_3, U_3). \tag{6}$$

Now by (5) and (6)

$$\mathbf{N}(x_1, U_1') \cap \mathbf{N}(x_2, U_2') \subset \mathbf{N}(x_1, U_1) \cap \mathbf{N}(x_2, U_2) \cap \mathbf{N}(x_3, U_3). \tag{7}$$

Since $x_1 \oplus x_2 \in FHC(T \oplus T)$, the set $\mathbf{N}(x_1, U_1') \cap \mathbf{N}(x_2, U_2')$ can be enumerated by a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of order $O(k)$. Thus there is a sequence $(n_k)_{k\in\mathbb{N}}$ and a constant $C$ such that for $i = 1, 2, 3$,

$$T^{n_k} x_i \in U_i \quad \text{and} \quad n_k \le Ck \quad \text{ for all } k \in \mathbb{N}$$

as desired. □

Now, our main result is followed by the induction.

**Theorem 2.3.** *Let $X$ be a separable $F$-space and let $T \in \mathcal{L}(X)$. If $T$ is frequently weakly mixing, then the $n$-fold product $T \oplus \cdots \oplus T$ is frequently hypercyclic.* □

**Remark 2.4.** In the proof of Theorem 2.2, we have used the sequence $(b_{k+N})$ which is closely related to the Hypercyclicity set defined in [3]. The set provides a relationship between the sets $\mathbf{N}(x, U)$ and plays an important role in testing whether an operator is weakly mixing or not. In our forthcoming paper, we will apply the ideas used in the proof of Theorem 2.2 to study the Hypercyclicity set in more detail. □

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